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Oscillation Theorems for Second-Order Half-Linear Differential Equations

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Abstract—Oscillation criteria for the second-order half-linear differential equation

$$[r(t)|x'(t)|^{\alpha-1}x'(t)]' + p(t)|x(t)|^{\alpha-1}x(t) = 0, \qquad t \geq t_0$$

are established, where $\alpha > 0$ is a constant and $\int_t^\infty p(s)\,ds$ exists for $t \in [t_0, \infty)$. We apply these results to the following equation:

$$\sum_{i=1}^{N} D_i(|Du(x)|^{n-2}D_iu(x)) + c(|x|)|u(x)|^{n-2}u(x) = 0, \qquad x \in \Omega_a,$$

where $D_i = \frac{\partial}{\partial x_i}$, $D = (D_1, \dots, D_N)$, $\Omega_a = \{x \in \mathbb{R}^N : |x| \geq a\}$ is an exterior domain, and $c \in C([a,\infty), \mathbb{R})$, $n > 1$ and $N \geq 2$ are integers. Here, $a > 0$ is a given constant.

Keywords—Half-linear differential equation, Oscillatory, Strong oscillation.

1. INTRODUCTION

Consider the following three second-order differential equations

$$[r(t)|x'(t)|^{\alpha-1}x'(t)]' + p(t)|x(t)|^{\alpha-1}x(t) = 0, \tag{E_1}$$
$$[\hat{r}(t)|x'(t)|^{\alpha-1}x'(t)]' + \hat{p}(t)|x(t)|^{\alpha-1}x(t) = 0, \tag{E_2}$$

and

$$\sum_{i=1}^{N} D_i\left(|Du(x)|^{n-2}D_iu(x)\right) + c(|x|)|u(x)|^{n-2}u(x) = 0, \qquad x \in \Omega_a, \tag{E_3}$$

where

(a) $p, \hat{p} \in C([t_0,\infty); \mathbb{R})$ for some $t_0 \geq 0$, $p(t) \not\equiv 0$, and $\hat{p}(t) \not\equiv 0$ on any interval of the form $[\tau, \infty)$ for $\tau \geq t_0$;
(b) $P(t) := \int_t^\infty p(s)\,ds$ and $\hat{P}(t) := \int_t^\infty \hat{p}(s)\,ds$ exist for $t \in [t_0, \infty)$;
(c) $r, \hat{r} \in C([t_0,\infty), (0,\infty))$ satisfy $\int^\infty r(s)^{-1/\alpha}\,ds = \infty$ and $\int^\infty \hat{r}(s)^{-1/\alpha}\,ds = \infty$;
(d) $D_i = \frac{\partial}{\partial x_i}$, $D = (D_1, \dots, D_N)$; $\Omega_a = \{x = (x_1, \dots, x_N) \in \mathbb{R}^N : |x| := \sqrt{x_1^2 + \cdots + x_N^2} \geq a\}$ is an exterior domain, and $c \in C([a,\infty), \mathbb{R})$ for some $a > 0$;
(e) $\alpha > 0$ is a constant, $n > 1$ and $N \geq 2$ are integers.

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By a solution of (E_1), we mean a function $x \in C[T_x, \infty)$, $T_x \geq t_0$, which has the property $|x'(t)|^{\alpha-1}x'(t) \in C^1[T_x, \infty)$ and satisfies (E_1) on $[T_x, \infty)$. In 1979, Elbert [1] established the existence and uniqueness of solutions to the initial value problem for equation (E_1) on $[t_0, \infty)$. We consider only those solutions $x(t)$ of (E_1) which satisfy $\sup\{|x(t)| : t \geq T\} > 0$ for all $T \geq T_x$. A nontrivial solution of (E_1) is called oscillatory if it has arbitrarily large zeros; otherwise, it is said to be nonoscillatory. Equation (E_1) is nonoscillatory (respectively, oscillatory) if all of its solutions are nonoscillatory (respectively, oscillatory).

Consider the linear differential equations

$$x''(t) + p(t)x(t) = 0, \tag{E_4}$$

and

$$(r(t)x'(t))' + p(t)x(t) = 0. \tag{E_5}$$

Surprisingly, some similar properties between equation (E_1) and the linear equation (E_4) or (E_5) have been observed by Elbert [1,2], Li and Yeh [3–6], and Mirzov [7–9]. For example, Sturmian theory for (E_4) has been extended in a natural way to (E_1) by Elbert [1] and Li and Yeh [3]. They showed that the zeros of two linearly independent solutions of (E_1) separate each other, and in particular, that all solutions of (E_1) are either oscillatory or nonoscillatory.

In 1987, Yan [10] gave some excellent oscillation criteria for equation (E_4) which extended some oscillation criteria of [11–18]. The purpose of this paper is to establish a necessary and sufficient condition for the nonoscillatory criterion of (E_1) which is a natural extension of Theorem 2.1 in [10]. Using this necessary and sufficient condition, we can extend the Hille-Wintner comparison theorem for equations of form (E_4) to equations of type (E_1). We apply these results to equations (E_3) by means of a method due to [19].

2. OSCILLATION CRITERIA FOR EQUATION (E_1)

In order to discuss our main results, we need two useful lemmas. The first one is a variant of Theorem 1 in [19] for equation (E_1) (see [12,17,20]) for $\alpha = 1$. For convenience's sake, a proof of this lemma is also provided.

LEMMA 1. *Suppose that (E_1) has a nonoscillatory solution $x(t) \neq 0$ on $[t_1, \infty)$ for some $t_1 \geq t_0$, and let*

$$w(t) = \frac{r(t)|x'(t)|^{\alpha-1}x'(t)}{|x(t)|^{\alpha-1}x(t)}.$$

Then

$$\int_t^\infty r(s)^{-1/\alpha}|w(s)|^{(\alpha+1)/\alpha}\,ds < \infty, \qquad \text{for } t \geq t_1, \tag{R_1}$$

$$\lim_{t\to\infty} w(t) = 0, \tag{R_2}$$

$$w(t) = \alpha\int_t^\infty r(s)^{-1/\alpha}|w(s)|^{(\alpha+1)/\alpha}\,ds + \int_t^\infty p(s)\,ds, \qquad \text{for } t \geq t_1. \tag{R_3}$$

PROOF. Without loss of generality, we may assume $x(t) > 0$ for $t \geq t_1 \geq t_0$; a similar argument holds if $x(t) < 0$ for $t \geq t_1 \geq t_0$.

It follows from (E_1) that the function

$$w(t) = \frac{r(t)|x'(t)|^{\alpha-1}x'(t)}{|x(t)|^{\alpha-1}x(t)}$$

satisfies

$$w'(t) + p(t) + \alpha r(t)^{-1/\alpha}|w(t)|^{(\alpha+1)/\alpha} = 0, \qquad \text{for } t \geq t_1. \tag{1}$$

Let $t \geq t_1$ be fixed arbitrary and integrate (1) over $[t, \tau]$:

$$w(\tau) - w(t) + \int_t^\tau p(s)\,ds + \alpha \int_t^\tau r(s)^{-1/\alpha}|w(s)|^{(\alpha+1)/\alpha}\,ds = 0, \qquad \text{for } \tau \geq t \geq t_0. \tag{2}$$

We claim that (R_1) holds. If

$$\int_t^\infty r(s)^{-1/\alpha}|w(s)|^{(\alpha+1)/\alpha}\,ds = \infty,$$

then it follows from (2) and (b) that there is a $T \geq t$ such that

$$\begin{aligned} w(\tau) + \alpha \int_T^\tau r(s)^{-1/\alpha}|w(s)|^{(\alpha+1)/\alpha}\,ds &= w(t) - \int_t^\tau p(s)\,ds - \alpha \int_t^T r(s)^{-1/\alpha}|w(s)|^{(\alpha+1)/\alpha}\,ds \\ &\leq -1, \end{aligned}$$

for $\tau \geq T$, or equivalently

$$-w(\tau) \geq 1 + \alpha \int_T^\tau r(s)^{-1/\alpha}|w(s)|^{(\alpha+1)/\alpha}\,ds, \qquad \text{for } \tau \geq T. \tag{3}$$

It follows that $x'(t) < 0$ for $\tau \geq T$ and

$$\frac{r(\tau)^{-1/\alpha}|w(\tau)|^{(\alpha+1)/\alpha}}{1 + \alpha \int_T^\tau r(s)^{-1/\alpha}|w(s)|^{(\alpha+1)/\alpha}\,ds} \geq r(\tau)^{-1/\alpha}[-w(\tau)]^{1/\alpha} = -\frac{x'(\tau)}{x(\tau)} \tag{4}$$

for $\tau \geq T$. Integrating (4) from T to τ, we get

$$\begin{aligned} \frac{1}{\alpha}\log\left[1 + \alpha \int_T^\tau r(s)^{-1/\alpha}|w(s)|^{(\alpha+1)/\alpha}\,ds\right] &= \int_T^\tau \frac{r(s)^{-1/\alpha}|w(s)|^{(\alpha+1)/\alpha}}{1 + \alpha \int_T^s r(t)^{-1/\alpha}|w(t)|^{(\alpha+1)/\alpha}\,dt}\,ds \\ &\geq -\int_T^\tau \frac{x'(s)}{x(s)}\,ds \\ &= \log\frac{x(T)}{x(\tau)}. \end{aligned}$$

Thus,

$$\left[1 + \alpha \int_T^\tau r(s)^{-1/\alpha}|w(s)|^{(\alpha+1)/\alpha}\,ds\right]^{1/\alpha} \geq \frac{x(T)}{x(\tau)}. \tag{5}$$

By (3)–(5), we obtain

$$-\frac{x'(\tau)}{x(\tau)} \geq \frac{x(T)}{x(\tau)} r(\tau)^{-1/\alpha},$$

therefore,

$$-x'(\tau) \geq x(T) r(\tau)^{-1/\alpha} \qquad \text{for } \tau \geq T.$$

Integrating the above inequality over $[T, \tau]$, we get

$$-\int_T^\tau x'(s)\,ds \geq x(T) \int_T^\tau r(s)^{-(1/\alpha)}\,ds.$$

This and (c) imply that $\lim_{\tau\to\infty} x(\tau) = -\infty$. This contradicts that $x(t) > 0$ on $[t_1, \infty)$. Therefore, (R_1) hold.

Letting $\tau \to \infty$ in (2) and using (R_4) and (b), we find that $w(\tau)$ tends to a finite limit $w(\infty)$. But $w(\infty)$ must be zero, since otherwise (R_1) would fail to hold. Thus, (R_2) and (R_3) hold. Hence, the proof of Lemma 1 is complete.

The following lemma is due to Li and Yeh [3].

LEMMA 2. *[3] Equation* (E_1) *is nonoscillatory if and only if there exists* $u \in C^1([t_1,\infty);\mathbb{R})$ *such that*

$$u'(t) + p(t) + \alpha r(t)^{-1/\alpha}|u(t)|^{(\alpha+1)/\alpha} \le 0.$$

Hereafter, we assume that $P(t) \ge 0$ and $\hat{P}(t) \ge 0$ on $[t_0,\infty)$. In order to discuss the oscillatory property for equations (E_1) and (E_2), we define the function sequences

$$\{\beta_n(t)\} \quad \text{and} \quad \left\{\hat{\beta}_n(t)\right\}, \qquad n = 0,1,2,\dots, \qquad \text{for } t \ge t_0 \tag{6}$$

as follows (if they exist):

$$\begin{aligned}
\beta_0(t) &= \int_t^\infty p(s)\,ds = P(t),\\
\hat{\beta}_0(t) &= \int_t^\infty \hat{p}(s)\,ds = \hat{P}(t),\\
\beta_n(t) &= \alpha \int_t^\infty r(s)^{-1/\alpha}\beta_{n-1}(s)^{(\alpha+1)/\alpha}\,ds + \beta_0(t), \qquad n = 1,2,\dots;\\
\hat{\beta}_n(t) &= \alpha \int_t^\infty \hat{r}(s)^{-1/\alpha}\hat{\beta}_{n-1}(s)^{(\alpha+1)/\alpha}\,ds + \hat{\beta}_0(t), \qquad n = 1,2,\dots.
\end{aligned}$$

Clearly, $\beta_1(t) \ge \beta_0(t)$ and $\hat{\beta}_1(t) \ge \hat{\beta}_0(t)$. By induction, we get

$$\beta_{n+1}(t) \ge \beta_n(t), \qquad n = 1,2,\dots; \tag{7}$$

$$\hat{\beta}_{n+1}(t) \ge \hat{\beta}_n(t), \qquad n = 1,2,\dots. \tag{8}$$

That is, the function sequences defined in (6) are nondecreasing on $[t_0,\infty)$.

Now we can state and prove one of our main results.

THEOREM 3. *Equation* (E_1) *is nonoscillatory if and only if there exists* $t_1 \ge t_0$ *such that*

$$\lim_{n\to\infty} \beta_n(t) = \beta(t) < \infty, \qquad \text{for } t \ge t_1. \tag{R_4}$$

PROOF. Suppose that $x(t)$ is a nonoscillatory solution of (E_1). Without loss of generality, we may assume that $x(t) > 0$ on $[t_1,\infty)$ for some $t_1 \ge t_0$. Let

$$w(t) = \frac{r(t)|x'(t)|^{\alpha-1}x'(t)}{|x(t)|^{\alpha-1}x(t)}.$$

Thus, it follows from Lemma 1 that (R_3) holds, and hence, $w(t) \ge \beta_0(t) \ge 0$. This implies

$$\begin{aligned}
w(t) &= \alpha \int_t^\infty r(s)^{-1/\alpha} w(s)^{(\alpha+1)/\alpha}\,ds + \beta_0(t)\\
&\ge \alpha \int_t^\infty r(s)^{-1/\alpha}\beta_0(s)^{(\alpha+1)/\alpha}\,ds + \beta_0(t)\\
&= \beta_1(t).
\end{aligned}$$

By induction,

$$w(t) \ge \beta_n(t) \ge 0, \qquad n = 0,1,2,\dots, \qquad t \in [t_1,\infty). \tag{9}$$

It follows from (7) and (8) that the function sequence $\{\beta(t)\}$ is bounded above on $[t_1,\infty)$. Hence, (R_4) holds.

Conversely, if (R_4) holds, then it follows from (7) and (R_4) that

$$\beta_n(t) \leq \beta(t), \qquad n = 0, 1, 2, \dots, \qquad \text{for } t \geq t_1.$$

Applying the Monotone Convergence Theorem,

$$\beta(t) = \alpha \int_t^\infty r(s)^{-1/\alpha} \beta(s)^{(\alpha+1)/\alpha} \, ds + \beta_0(t).$$

Clearly, $\beta(t) \in C^1$ and

$$\beta'(t) + p(t) + \alpha r(t)^{-1/\alpha} \beta(t)^{(\alpha+1)/\alpha} = 0.$$

It follows from Lemma 2 that (E_1) is nonoscillatory. Thus, our proof is complete.

Clearly, we have the following corollary.

COROLLARY 4. *Equation (E_1) is oscillatory if and only if either*

(i) *there exists a positive integer m such that $\beta_n(t)$ is defined for $n = 1, 2, \dots, m-1$, but $\beta_m(t)$ does not exist; or*

(ii) *$\beta_n(t)$ is defined for $n = 1, 2, \dots$, but for arbitrarily large $T \geq t_0$, there is $t^* \geq T$ such that*

$$\lim_{n\to\infty} \beta_n(t^*) = \infty.$$

Using Theorem 3, we can give another method to prove the following Hille-Wintner comparison theorem (see [6,19]).

THEOREM 5. *Assume that*

$$0 < \hat{r}(t) \leq r(t), \qquad P(t) \leq \hat{P}(t), \qquad \text{for all sufficiently large } t. \tag{10}$$

If (E_2) is nonoscillatory, then (E_1) is nonoscillatory.

PROOF. Suppose that (E_2) is nonoscillatory. It follows from Theorem 3 that there exists $t_1 \geq t_0$ such that

$$\lim_{n\to\infty} \hat{\beta}_n(t) = \hat{\beta}(t) < \infty, \qquad \text{for } t \geq t_1. \tag{11}$$

Clearly, by (10),

$$0 \leq \beta_0(t) := P(t) \leq \hat{P}(t) := \hat{\beta}_0(t).$$

This implies

$$\begin{aligned}
\beta_1(t) &= \alpha \int_t^\infty r(s)^{-1/\alpha} \beta_0(s)^{(\alpha+1)/\alpha} \, ds + \beta_0(t) \\
&\leq \alpha \int_t^\infty \hat{r}(s)^{-1/\alpha} \hat{\beta}_0(s)^{(\alpha+1)/\alpha} \, ds + \hat{\beta}_0(t) \\
&= \hat{\beta}_1(t).
\end{aligned}$$

By induction,

$$0 \leq \beta_n(t) \leq \hat{\beta}_n(t), \qquad n = 0, 1, 2, \dots, t \in [t_1, \infty). \tag{12}$$

Therefore, by (8), (11), and (12),

$$\beta_n(t) \leq \hat{\beta}(t) < \infty, \qquad n = 0, 1, 2, \dots, t \in [t_1, \infty).$$

This and (7) imply

$$\lim_{n\to\infty} \beta_n(t) = \beta(t) < \infty, \qquad \text{for } t \geq t_1 \geq t_0.$$

Thus, by Theorem 3, (E_1) is nonoscillatory. Hence, the proof is complete.

We note that Theorems 3 and 5 and Corollary 4 also hold if the function sequences in (6) are defined by

$$
\begin{aligned}
\beta_0(t) &= \int_t^\infty p(s)\,ds = P(t),\\
\hat\beta_0(t) &= \int_t^\infty \hat p(s)\,ds = \hat P(t),\\
\beta_1(t) &= \alpha\int_t^\infty r(s)^{-1/\alpha}\beta_0(s)^{(\alpha+1)/\alpha}\,ds,\\
\hat\beta_1(t) &= \alpha\int_t^\infty \hat r(s)^{-1/\alpha}\hat\beta_0(s)^{(\alpha+1)/\alpha}\,ds,\\
\beta_{n+1}(t) &= \alpha\int_t^\infty r(s)^{-1/\alpha}[\beta_n(s)+\beta_0(s)]^{(\alpha+1)/\alpha}\,ds, \qquad n=1,2,\dots.\\
\hat\beta_{n+1}(t) &= \alpha\int_t^\infty \hat r(s)^{-1/\alpha}\left[\hat\beta_n(s)+\hat\beta_0(s)\right]^{(\alpha+1)/\alpha}\,ds, \qquad n=1,2,\dots.
\end{aligned}
$$

3. OSCILLATION CRITERIA FOR EQUATION (E_3)

In this section, we are interested in radial solutions of (E_3); that is, those solutions which depend only on $|x|$. It is easy to verify that a function $u = y(|x|)$ is a solution of (E_3) in Ω_a if and only if $y(t)$ is a solution of the ordinary differential equation

$$
\left(t^{N-1}|y'|^{n-2}y'\right)' + t^{N-1}c(t)|y|^{n-2}y = 0, \qquad t \ge a. \tag{E_6}
$$

This equation is a special case of (E_1) in which $\alpha = n-1$, $r(t) = t^{N-1}$, and $p(t) = t^{N-1}c(t)$. Note that condition (c) holds for (E_6) if and only if $n \ge N$. The function sequences $\{\beta_n(t)\}$ defined as in (6) are given as follows:

$$
\begin{aligned}
\beta_0(t) &= \int_t^\infty s^{N-1}c(s)\,ds = P(t),\\
\beta_n(t) &= (n-1)\int_t^\infty \left(s^{N-1}\right)^{-1/(n-1)}\beta_{n-1}(s)^{n/(n-1)}\,ds + \beta_0(t), \qquad n=1,2,\dots.
\end{aligned}
$$

Applying the results of the preceding section to (E_1), we obtain the following two results.

THEOREM 6. *Let $P(t) \ge 0$ on $[t_0,\infty)$. Then all radial solutions of (E_3) are nonoscillatory if and only if there exists $t_1 \ge t_0$ such that*

$$
\lim_{n\to\infty}\beta_n(t) = \beta(t) < \infty, \qquad \text{for } t \ge t_1.
$$

COROLLARY 7. *Let $P(t) \ge 0$ on $[t_0,\infty)$. Then all radial solutions of (E_3) are oscillatory if and only if either*

(i) *there exists a positive integer m such that $\beta_n(t)$ is defined for $n = 1,2,\dots,m-1$, but $\beta_m(t)$ does not exist; or*
(ii) *$\beta_n(t)$ is defined for $n = 1,2,\dots$, but for arbitrarily large $T \ge t_0$, there is $t^* \ge T$ such that*

$$
\lim_{n\to\infty}\beta_n(t^*) = \infty.
$$

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